AN EXTENSION OF GREEN'S LEMMA TO THE CASE OF A RECTIFIABLE BOUNDARY.*

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The current proofs of the fundamental Green's Lemma,

$$\int\int_K \frac{\partial P(x, y)}{\partial x}\, dydx = \int_C P(x, y)dy, \tag{1}$$

introduce notable restrictions upon the boundary C of K, the usual restriction being the hypothesis that the boundary is cut by a parallel to the axis of x in only a finite number of points. So far as I have ascertained,† no proof has been previously given covering the general case in which the boundary is merely restricted by the requirement that it shall be rectifiable. Such a proof is given in this note, and furthermore, no condition is imposed upon $P(x, y)$ except its continuity and the integrability of its derivative $\partial P/\partial x$ over the two-dimensional field K.

Cauchy's basal theorem for an analytic function,

$$\int_C f(z)dz = 0,$$

then follows in the usual way, being thus extended to the case of a rectifiable boundary. It should, however, be remarked that this theorem is also established in Jordan's Cours d'analyse‡ under the hypothesis of a rectifiable boundary but independently of Green's Lemma.

For simplicity of presentation the proof of Green's Lemma is first given for a simply connected region bounded by a *simple* closed rectifiable curve, that is, one without double points. The unessential restriction of a boundary without double points and of a region with simple connection is then quickly removed. The method of proof adopted is capable of extension to corresponding theorems in three or more dimensions.

Consider a region K of the plane bounded by a *simple* closed rectifiable curve C:

$$x = \phi(t), \qquad y = \psi(t) \qquad (t_0 \leqq t \leqq t'; \ \phi(t_0) = \phi(t'), \ \psi(t_0) = \psi(t')).$$

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† For extensions of a different nature see Cino Poli, Atti della r. Academia delle Scienze di Torino, vol. 49 (1913–4), pp. 248–258, and M. Picone, Rendiconti del Circolo Matematico di Palermo, vol. 43 (1918–9), p. 239.

‡ Vol. 1, second edition, pp. 185–191.

For convenience we suppose that a point O within the curve is taken as the origin of coördinates, also that the curve is described in a positive direction when t passes from t_0 to t'. Inclose C in a rectangle with sides parallel to the axes. Then subdivide the rectangle by means of two systems of parallels,

$$x = m/2^n, \qquad y = m'/2^n \qquad (m, m' = 0, \pm 1, \pm 2, \cdots).$$

Thereby it is divided into squares of side $1/2^n$. If n is a sufficiently large integer, O will belong to four initial squares, all of whose points *inclusive of their boundary points* are interior to C. To these initial squares we will annex any other square which borders one of the initial squares along a side and which is entirely free from points of C. If one of the annexed squares is similarly bordered by a square entirely free from points of C, we will annex this square and continue the annexation until it is no longer possible to annex squares entirely free from points of C. In this manner we obtain a connected region consisting only of points interior to C, which we will call the *checkerboard net* K_n belonging to the origin O for a given value of n. Obviously K_n is included in K_{n+1}.

This checkerboard net is a *simply connected* region bounded by a *simple* polygon whose sides are parallel to the axes. To see this, trace any boundary line of the checkerboard net until it first returns upon itself at some point P. The portion included between two successive passages through P is a simple closed polygon whose sides are parallel to the axes and consist of segments of length $1/2^n$. Clearly the interior of this polygon, like the perimeter itself, is interior to C. Now each segment of the perimeter separates a square S' of the checkerboard net containing only points interior to C from a square S'' which contains at least one point of C. The squares inside the polygon therefore belong to the checkerboard net. Since also no square exterior to the polygon can be connected with one in the interior by a chain of contiguous squares without including an S'', this polygon is the whole of the checkerboard net K_n. The latter is accordingly a simply connected region bounded by a simple polygon, as stated.

It can next be shown that when n is sufficiently increased, K_n may be made to include any assigned point P interior to C. For let P be connected with O by some continuous curve OP lying wholly within C. Denote with δ the minimum distance between points of OP and points of C, and increase n so that the length of the diagonals of the squares composing K_n shall be smaller than δ. Then any point of OP will either lie within a square containing no points of C or lie on a common side of two such squares or be the vertex common to four such squares. The line OP must therefore lie within K_n.

Since C is rectifiable, it is also squarable; in other words, the total area of our squares containing points of C can be made smaller than a prescribed ϵ by sufficiently increasing n. Denote with A the area within C. The total area of all squares wholly interior to C will be greater than $A - \epsilon$. For fixed $n = N$ this total area may consist of a number of separate pieces which are composed of squares of side $1/2^N$. If a point P is selected in any one of these component squares, it will be included within K_n when n is sufficiently increased, and simultaneously with P the whole of the component square. Hence all the squares composing our fixed area will ultimately be included in K_n. Consequently by sufficiently increasing n the area of K_n may be made to differ from that within C by less than ϵ.

Consider now a function $P(x, y)$ which is continuous over the closed field K bounded by C and has a derivative $\partial P(x, y)/\partial x$ which is properly integrable over this field. By a well-known theorem* the existence of the proper double integral $\int\int(\partial P/\partial x)dxdy$ over a rectangular field with sides parallel to the axes carries with it the existence of the two iterated integrals and their equality to the double integral. Now this iterated integral $\int dy \int(\partial P/\partial x)dx$ is equal to the curvilinear integral $\int P(x, y)dy$ taken in a positive direction around the boundary, the components of this curvilinear integral over the horizontal sides of the rectangle being, of course, equal to zero. Hence Green's Lemma (1) holds for each square of our checkerboard net. By addition of the squares we obtain

$$\int\int_{K_n} \frac{\partial P}{\partial x}\, dxdy = \int_{C_n} P(x, y)dy, \tag{2}$$

where C_n denotes the boundary of K_n.

The remainder of our work consists of the extension of Green's Lemma from K_n to the field K by passing to the limit with indefinitely increasing n.

By hypothesis $\partial P/\partial x$ is properly integrable over K and has therefore an upper limit M to its absolute value. By sufficiently increasing n the difference of the areas of K and K_n can be made smaller than a prescribed ϵ. Then the left hand member of (2) will differ from that of (1) by less than ϵM. Consequently the left hand member of (2) approaches the left-hand member of (1) as its limit when n is indefinitely increased.

Before making a comparison of the right-hand members of (1) and (2) it may be remarked that the former is given its natural meaning and is accordingly a Stieltjes integral obtained as follows. Let the interval $(t_0, t' = t_{n+1})$ be subdivided by successive values $t_1, t_2, \cdots, t_k, \cdots, t_n$, and denote by $R_k = (x_k, y_k)$ the corresponding points of C. Form the sum

$$\sum_{k=0}^{n} P(\xi_k, \eta_k)\cdot(y_{k+1} - y_k), \tag{3}$$

* Cf. Hobson, The Theory of Functions of a Real Variable, § 314, p. 425.

in which (ξ_k, η_k) denotes a point taken arbitrarily on C between R_k and R_{k+1} inclusive. If this sum tends to a limit when n increases and the maximum size (norm) of the subintervals $|t_{k+1} - t_k|$ is indefinitely decreased, the limit is the integral denoted by the right hand member of (1). This limit exists when $P(x, y)$ is continuous and $\psi(t)$ is a function of limited variation.* In the case before us the variation

$$\sum_{k=0}^{n} |\psi_{k+1}(t) - \psi_k(t)| \equiv \sum_{k=0}^{n} |y_{k+1} - y_k|$$

is limited because this sum is less than the length L of our rectifiable curve C.

Suppose that the t-norm has been taken so small that the difference between the approximation (3) and the right-hand member of (1) is less in absolute value than an assigned ϵ. *Without infringing this requirement* we may modify the partition $R_0R_1R_2 \cdots R_{n+1}$ of C and correspondingly the sum (3) in the following manner. If any division R_kR_{k+1} of C is a horizontal segment, the corresponding term in (3) is zero since $y_{k+1} = y_k$. Each succession of such horizontal divisions may therefore be merged into a single horizontal division R_kR_{k+1} without changing the value of (3). If then an adjoining piece $R_{k+1}R''$ of the division following (similarly of the division just preceding) is also a horizontal segment, we will interpolate between t_{k+1} and t_{k+2} a division point t'' corresponding to R'', modifying at the same time the sum (3) correspondingly. Then the horizontal piece $R_{k+1}R''$ can be merged with R_kR_{k+1}. In this manner let the horizontal divisions of our partition be made of maximum length. Each horizontal division R_kR_{k+1} is then followed (and similarly preceded) by a division in which there are points (x, y) as near to R_{k+1} as we please for which $|y - y_{k+1}| \neq 0$. Our partition of C is then *normalized.*

We now proceed to partition the boundary of our checkerboard net. Start from any corner P_0 of K_n and trace its boundary C_n in a positive direction. As we pass successively through vertices of component squares of K_n, denote these vertices by $P_1, P_2, P_3, \cdots$, omitting in this enumeration all vertices on each horizontal side of C_n except its extremities. The vertical sides of C_n are thus divided into segments of length $1/2^n$, while the horizontal sides are multiples of this length. (Fig. 1.)

To this partition of C_n we will make correspond a partition $Q_0Q_1Q_2 \cdots$ of C in the following manner. Every line $y = m/2^n$ which crosses or bounds K_n ceases to cross or bound it at a point P_i. We will take as the corresponding point Q_i the first point in which the line meets C after emergence at P_i. (See Fig. 1.) A division point Q_i of C is thus obtained

* Cf. Vallée Poussin, Cours d'analyse infinitésimale, vol. 1, 3d ed., § 346.

for every P_i except those at which the interior of K_n makes a 270° bend. Suppose, if possible, the line to meet C_n between P_i and Q_i, and let P' be the first point of meeting. Then P_iP' and one or the other of the two portions of C_n between P_i and P' would together form a simple closed polygon whose interior is entirely free from points of C and should therefore belong to the checkerboard net, whereas it lies without. It follows that the interior of the horizontal segment P_iQ_i is exterior to K_n. Any

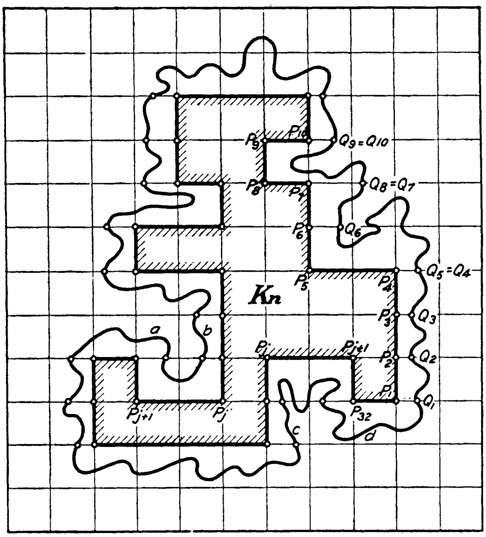

FIG. 1.

two such lines P_iQ_i and $P_{i+k}Q_{i+k}$ with the included part

$$P_iP_{i+1}P_{i+2} \cdots P_{i+k}$$

of C_n and the corresponding portion of C will bound a simply connected region interior to C and exterior to K_n. Into this region the lines $P_{i+j}Q_{i+j}$ $(j < k)$ penetrate. Since the points Q_{i+j} are the points where these lines first meet C, they must be included on C between Q_i and Q_{i+k}. It follows that *the order of the Q_i on C is exactly the same as the order of the corresponding P_i on C_n.*

It remains yet to fix a point Q_i on C corresponding to a vertex P_i of K_n at which the angle of the polygon is 270°. If at the other extremity of the horizontal side which terminates in such a P_i the angle is not 270°, we have already fixed a point Q corresponding to that extremity. We will then make this same point correspond also to P_i so that we have either $Q_{i-1} = Q_i$ or $Q_{i+1} = Q_i$. On the other hand, when the angles at both extremities of a horizontal side P_jP_{j+1} are 270°, we will slightly modify

our polygon K_n in order that the corresponding points Q_j, Q_{j+1} on C shall have the same ordinate as P_j and P_{j+1}. According as the exterior of K_n lies just above or below P_jP_{j+1}, we will move this side parallel to itself upward or downward until it first contains a point of C (see Fig. 2). The amount of this vertical displacement can not exceed $1/2^n$. The alteration in K_n merely increases its area and therefore does not affect the validity of our previous comparison between the double integrals (1) and (2). We will now take as our points Q_j, Q_{j+1} the points* of C

FIG. 2.

on the new and displaced side P_jP_{j+1} which are nearest to P_j and P_{j+1} respectively. Thus Q_j, Q_{j+1} are inserted in order on C between Q_{j-1} and Q_{j+1}.

It will next be shown that the Q_i remain partition points of C when n is increased by 1. The segments P_iQ_i were, in fact, the portions of the lines $y = m/2^n$ which are included between C_n and C. Now these lines are included among the lines $y = m/2^{n+1}$. Since K_{n+1} contains K_n, the segment P_iQ_i is merely replaced by the portion of the segment included between K_{n+1} and C when n is replaced by $n + 1$. Hence the Q_i continue to be partition points when n is increased. To see that this is also true of Q_j and Q_{j+1}, consider the squares exterior to the (unmodified) K_n which border it along P_jP_{j+1} before displacement of this side. Since these are of side $1/2^n$, each is divided into four subsquares when n is increased by 1. Some of these subsquares may be annexed to K_n to form K_{n+1}. If Q_j, Q_{j+1} lie in subsquares adjacent to P_jP_{j+1}, it is impossible to annex all the adjacent subsquares. Hence either the whole of P_jP_{j+1} or portions of the same are included as sides in the boundary of K_{n+1}. The angles of K_{n+1} at the extremities of these sides are obviously 270°, and Q_j, Q_{j+1} are the points of C nearest to one or two of these sides. Thus Q_j, Q_{j+1} play the same rôle relatively to K_{n+1} as to K_n. On the other hand, if none of the subsquares adjacent to P_jP_{j+1} contain either Q_j or Q_{j+1}, all these subsquares may be annexed to K_n, whereby P_jP_{j+1} is displaced by an amount $1/2^{n+1}$. Then Q_j and Q_{j+1} lie in subsquares adjacent to the displaced P_jP_{j+1} and the situation is exactly the same as that just described. Accordingly Q_j, Q_{j+1} have the same relation to the unmodified K_{n+1} as

* The two points may, of course, coincide. It may be further remarked that when the displacement of the side is exactly $1/2^n$, we have a coincidence of P_j, P_{j+1} with P_{j-1}, P_{j+2} and correspondingly of Q_j, Q_{j+1} with Q_{j-1}, Q_{j+2}.

to the unmodified K_n, and they must therefore be retained as division points of C when n is increased.

Parenthetically it may be remarked that our final checkerboard net K_n, obtained through modification in the manner above described, is included in the modified K_{n+1}.

We will next see that *when n is sufficiently increased, the division points Q_i enter into any portion of C which is not a horizontal segment.* This is true, for example, even if horizontal segments are everywhere dense in the portion considered. For proof, take any two points Q', Q'' with different ordinates in the portion under consideration; and let $m/2^n$ be an intermediate ordinate. Draw the line $y = m/2^n$ and suppose Q' to lie below this line, Q'' above. A sufficiently small vicinity of any point of C below the line can not contain any points interior to K which are above the line. As we pass along C from Q' to Q'', we must come either to a last point of this nature or to a first point $\bar{Q}$ whose vicinity, no matter how small, will contain interior points of K above the line. The former alternative is impossible since the point would be the limit of points with vicinities of both characters, and any vicinity of the point would therefore contain interior points of K both above and below the line. Clearly $\bar{Q}$ lies on the line $y = m/2^n$. Consider now an interval of the line having $\bar{Q}$ as its center and a length δ less than the minimum distance from $\bar{Q}$ to any point of C not included in the portion $Q'Q''$. We will establish the intuitive fact that the interior of K must cut across the interval. Suppose, if possible, that it does not. Describe a circle of radius $\delta/2$ about $\bar{Q}$ as center. The semi-perimeter of this circle below the interval is crossed one or more times by the interior of K since any vicinity of $\bar{Q}$ contains points interior to K and below $y = m/2^n$. Take a piece $A\bar{Q}B$ of the arc $Q'\bar{Q}Q''$ of C with a length less than $\delta/2$, A being supposed to lie between Q' and $\bar{Q}$. This piece will contain all points of C sufficiently near to $\bar{Q}$ since C is by hypothesis without double points. Hence $A\bar{Q}$ will be part of the boundary of one of the pieces of the interior of K which penetrate into the semicircle below our interval. Then $\bar{Q}B$ is the continuation of its boundary. Since by hypothesis the interior of this piece does not cross the interval, it follows that all points of $\bar{Q}B$ lie on or below the line $y = m/2^n$. Now no other portion of K exterior to this piece can come within a sufficiently small vicinity of $\bar{Q}$ since C is without double points. Consequently in a sufficiently small vicinity of $\bar{Q}$ there can not lie any points of K above $y = m/2^n$, which gives a contradiction. We conclude therefore that the interior of K crosses the δ-interval, as stated, and cuts out one or more subintervals.

Consider any one of these subintervals. Let J be an interior point. When n is sufficiently increased, J will be included within our checker-

board net K_n. The vector $J\bar{Q}$ emerges then from K_n between J and $\bar{Q}$, and the first point in which it meets C after emergence will be a division point Q_i of C. This point lies in $Q'QQ''$. Hence the points Q_i will enter into any portion of C which is not a horizontal segment when n is sufficiently increased.

Our partitions $P_0P_1P_2 \cdots$ and $Q_0Q_1Q_2 \cdots$ of C_n respectively were so formed that corresponding points P_i and Q_i have the same ordinate y_i. With respect to these partitions we will now form approximating sums for $\int_{C_n} P(x, y)dy$ and $\int_C P(x, y)dy$. To prove then that the latter integral is the limit of the former when n is increased indefinitely, it will suffice to show that the difference between these two approximations, and between each approximation and the corresponding integral, can be made arbitrarily small by sufficiently increasing n.

Construct first with respect to C_n the sum

$$\sum_i P(\xi_i', \eta_i') \cdot (y_{i+1} - y_i), \tag{4}$$

in which y_{i+1}, y_i are the ordinates of P_{i+1}, P_i and (ξ_i', η_i') is an arbitrarily chosen point of P_iP_{i+1}. In this sum all terms vanish except those which relate to vertical segments P_iP_{i+1} of C_n. These are of length $1/2^n$. Now the quadrilateral bounded by the linear segments P_iP_{i+1}, P_iQ_i, $P_{i+1}Q_{i+1}$ and the arc Q_iQ_{i+1} of C is exterior to K_n and contains in its interior no points of C. Also in the checkerboard division of K each vertical segment P_iP_{i+1} of K_n borders a square exterior to K_n which contains at least one point of C. This can only be if the arc Q_iQ_{i+1} enters or touches the square and accordingly contains at least one point (ξ_i, η_i) in the square. Take such a point (ξ_i, η_i) for every arc Q_iQ_{i+1} corresponding to a vertical P_iP_{i+1} and an arbitrary point of the arc when $y_i = y_{i+1}$, and construct the sum

$$\sum_i P(\xi_i, \eta_i) \cdot (y_{i+1} - y_i). \tag{5}$$

Since $P(x, y)$ is supposed continuous over K, the variation of $P(x, y)$ can be made less than an arbitrarily assigned ϵ simultaneously in every checkerboard divisions of K by making the norm $1/2^n$ sufficiently small, i.e., by making n sufficiently large. Then the difference of the approximations (4) and (5) is numerically less than $\epsilon\Sigma_i |y_{i+1} - y_i|$. Since $|y_{i+1} - y_i|$ is the vertical distance between two consecutive divisions points Q_i, Q_{i+1} of C, this quantity is less than ϵL, where L denotes the length of C. Furthermore, by the same uniform continuity each term

$$P(\xi_i', \eta_i') \cdot (y_{i+1} - y_i)$$

in (4) will differ from the value of $\int_{y_i}^{y_{i+1}} P(x, y)dy$, taken along C_n between P_i and P_{i+1}, by less than $\epsilon|y_{i+1} - y_i|$. Consequently the values of (4) and of $\int_{C_n} P(x, y)dy$ can also be made to differ by less than the arbitrary quantity ϵL by sufficiently increasing n.

It remains now only to prove that the sum (5) can be made to differ from $\int_C P(x, y)dy$ by as little as we wish. To establish this, let us compare (5) with an approximation (3) for the latter which corresponds to a partition $R_0R_1R_2 \cdots$ of C normalized in the manner previously described. We will suppose that the t-norm for the partition is taken so small that the variation of $P(x, y)$ in each R_kR_{k+1} (with the exception of horizontal divisions) is less than an assigned ϵ'', and also so small that the approximation (3) differs from $\int_C P(x, y)dy$ as little as we wish. Denote by ν the number of horizontal divisions R_kR_{k+1} which are horizontal segments. If now, as we shall suppose, n has been taken sufficiently large, the Q_iQ_{i+1} will be found in every R_kR_{k+1} except the ν horizontal divisions, and, furthermore, the distances along C from the terminal points R_k, R_{k+1} of the horizontal divisions to the first Q_i in the divisions respectively preceding and following may be supposed less than an arbitrarily assigned ϵ'. To facilitate comparison between (3) and (5), put in (3)

$$y_{k+1} - y_k = (y_i - y_k) + (y_{i+1} - y_i) + \cdots + (y_{k+1} - y_{i+j}),$$

when points Q_i, Q_{i+1}, $\cdots$, Q_{i+j} are contained in R_kR_{k+1}. Similarly write

$$y_{i+1} - y_i = (y_k - y_i) + (y_{i+1} - y_k),$$

when Q_i, and Q_{i+1} are separated on C by a R_k, and

$$y_{i+1} - y_i = (y_k - y_i) + (y_{k+1} - y_k) + (y_{i+1} - y_{k+1}),$$

when separated by a horizontal segment R_kR_{k+1}. We have thus split up (3) and (5) for comparison into an equal number of terms corresponding to the same subdivisions of C.

These subdivisions are of four sorts. First, there are the ν horizontal divisions of the R-partition. The corresponding terms in (3) and (5) vanish since the ordinates of the extremities of such a division are equal. Secondly, there are 2ν adjoining subdivisions, each less than ϵ' in length. If M denotes the maximum of the absolute value of $P(x, y)$ upon C, the sum of the 2ν components in (3) or in (5) corresponding to these subdivisions will not exceed $2\nu\epsilon' M$ in absolute value. Thirdly, we have as

subdivisions such of the Q_iQ_{i+1} as lie each entirely in some R_kR_{k+1}. The corresponding terms in (3) and in (5) will then not differ by as much as $\epsilon''|y_{i+1} - y_i|$. Lastly, we have subdivisions Q_iR_k and R_kQ_{i+1} due to separation of Q_i and Q_{i+1} by an R_k. Since the multipliers of $y_k - y_i$ and $y_{i+1} - y_k$ in (5) are values of $P(x, y)$ taken at points of Q_iQ_{i+1} and hence taken from the same or adjacent divisions $R_{k-1}R_k$ and R_kR_{k+1}, they differ from the corresponding multipliers in (3) by less than $2\epsilon''$. It follows that the values of (3) and (5) differ by less than

$$4\nu\epsilon'M + \epsilon''\Sigma|y_{i+1} - y_i| + 2\epsilon''\Sigma(|y_k - y_i| + |y_{i+1} - y_k|)$$
$$< 4\nu\epsilon'M + \epsilon''(L + 2L).$$

Now we first prescribed ϵ'' and then *fixed* an R-partition with some value of ν. Then by sufficiently increasing n we made ϵ' and therefore $\epsilon'\nu$ as small as we please. Hence by prescribing ϵ'' and increasing n we may make (3) and (5) to differ as little as we choose. Since also (3) was chosen arbitrarily near to $\int_C P(x, y)dy$, we conclude that when n is indefinitely increased, the sum (5) approaches this integral as its limit. This completes the proof that the limit of the right-hand member of (2) is the right-hand member of (1) and establishes Green's lemma for the case of a simple rectifiable curve.

Extension can now be quickly made to the general case of a simply connected region bounded by a rectifiable curve C. Multiple points of C may exist, and portions of it may be bordered by the interior of K on opposite sides, being twice traced and in opposite directions when C is completely described. The preceding proof applies except for the two paragraphs which show that the Q_i enter with increasing n into any portion of C which is not a horizontal segment. To establish this we may make use of the fact that when the boundary of a simple connected region is the continuous image of a circle, all of its points are "*attainable.*"* In other words, any point of the boundary can be reached from any interior point by a polygonal line lying entirely in the interior (except for its end point on the boundary) or approached as limit point by a polygonal line consisting of an infinite number of interior segments. Since C is rectifiable and continuous, it is the continuous image of a circle, and all of its points are consequently attainable from the interior of K.

Consider now any portion of C which is not a horizontal segment and, as before, take on it any two points Q', Q'' with different ordinates. From

* Schoenflies, Die Entwickelung der Lehre von den Punktmannigfaltigkeiten, Zweiter Teil, p. 189; Jahresbericht der deutschen Mathematiker Vereinigung, 1908.

If the correspondence between the boundary and circle is also one-to-one, the former is a simple closed curve.

any point S in the interior of K draw non-intersecting polygonal lines SQ' and SQ'' which "attain" to Q' and Q'' as end- or limit-points. These two lines together, like a cross cut, divide the interior into two separate regions. For if there remains a connected whole region, we could draw in its interior a simple closed polygon starting at S on one side of the lines SQ', SQ'' and terminating at S on the other side. One of these lines, entering the interior of the polygon would have to remain in it, which contradicts the hypothesis that they attain to Q' and Q'' respectively.

Consider now the region bounded by $Q'SQ''$ and the arc $Q'Q''$ of C under consideration. Let $m/2^n$ be any ordinate intermediate between those of Q' ,Q'' and draw as above the line $y = m/2^n$. The interior of our region cuts out a finite or countably infinite number of intervals on this line. We will show that at least one of these intervals has an end point on the arc $Q'Q''$. Suppose, if possible, the contrary. Then every interval, considered as a crosscut, divides the region into two parts, for one of which the arc $Q'Q''$ is a part of the boundary while for the other it is not. Any interior point of the region whose minimum distance from the arc $Q'Q''$ is less than that to $Q'SQ''$ must obviously lie in the former of the two parts. Draw now in the interior of the region a polygonal path connecting two such points A and B, the former of which lies below $y = m/2^n$ and the other above. This path crosses any one of the above intervals only a finite number of times. Suppose it to cross an infinite number of the intervals. As we are concerned only with a limited portion of the plane, this infinite set of intervals on $y = m/2^n$ must have the lower limit zero to their length, and there must be at least one limit point for the set. A limit point, on the one hand, must lie on $Q'SQ''$, being the limit of end-points of our intervals which themselves lie upon it; on the other hand, since the continuous path approaches to within an infinitesimal distance of the limit point, this point must lie on the path and hence be an interior point of the region. From this contradiction it follows that the path can cross only a finite number of our intervals. Now if in passing from A to B the path crosses any interval, it must ultimately cross back again, for otherwise it would remain in the subregion bounded by the interval and the included piece of $Q'SQ''$ and hence could not reach B. Replace the portion of the path between the first point of crossing an interval and the last point of crossing the same interval by the segment of the interval between these two points. If the modified path still crosses an interval, do the same thing for the first remaining interval it crosses, and so on. We have finally a path which does not cross any one of our intervals and thus connects A and B on opposite sides of the line $y = m/2^n$ without crossing the line. Since this is impossible, our hypothesis is

untenable. We conclude therefore that some one of our intervals has an end point Q on the arc $Q'Q''$ of C.

We may now argue precisely as in the case of a simple rectifiable boundary that this point Q ultimately becomes a division point Q_i of C. If, namely, we taken an interior point $\bar{Q}$ of the interval which ends in Q, then $\bar{Q}$ will become a point of our checkerboard net when n is sufficiently increased. Since Q is the first point in which $y = m/2^n$ meets C after an emergence from the net, we conclude that Q is now a division point Q_i. The remainder of our proof then applies as before.

Extension can be readily made to the case of a multiple connected region bounded by n rectifiable curves, over which $P(x, y)$ is uniquely defined. For this purpose the region can be rendered simply connected by drawing rectilinear cuts between pairs of boundaries. These cuts are without effect upon the right and left hand members of (1).

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